Experiencing the Multiple Dimensions of Mathematics with Dynamic 3D geometry Environments: Illustration with Cabri 3D

Colette Laborde
Colette.Laborde@imag.fr
LIG, University Joseph Fourier FRANCE

Abstract. *The paper analyzes how 3D dynamic geometry environments may be used to foster the exploration of multiple dimensions of 3D geometry. The notion of dimension is twofold: it refers, one the one hand to the dimension of a geometrical object, on the other hand to the multiple types of representation and expressions used in geometry. Two kinds of processes are involved in problem solving in geometry: iconic and non iconic visualization. The non iconic visualization consists in breaking down an object into parts of same or lower dimension. This cognitive process is critical for solving problems in geometry as very often the reasoning consists in establishing relationships between elements of the figure. However this process is not spontaneous and must be learned. 3D geometry is the source of new problems regarding iconic and non iconic visualization. Iconic visualization is not always reliable as it is in 2D geometry and non iconic visualization is more complex since it deals with a larger number of kinds of objects, from dimension 0 to dimension 3. The paper examines how 3D dynamic geometry environments may enlarge the iconic visualization and assist the non iconic visualization. 3D geometry computer environments may also offer a textual description linked to the dynamic diagram. The interplay between both representations not only facilitates the construction process of figures but also may be used to move from construction tasks to proof tasks. The example of Cabri 3D is used in the paper to illustrate the argument.*

1. Foreword

At first glance, the notion of dimension refers to the dimension of geometrical objects such as point (dimension 0), line (dimension1), and plane (dimension 2). But we also want to refer in this presentation to external representations or registers (in terms of Duval 2000) used in mathematics for representing mathematical objects and relations or describing them, such as graphical representations like diagrams in paper and pencil or on a computer screen, or natural language.

2. “Deconstructing” and structuring 3D objects

2.1. Why is it difficult to count the number of faces of a polyhedron?

Imagine a middle school or high school student faced with the task of counting the faces of a real solid object without the possibility of marking these faces. Suppose that it is a rhombic dodecahedron (Fig.1).

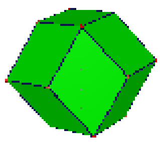


Figure 1 - A rhombic dodecahedron[1] rhombicdodecahedron.cg3

[1]Cabri 3D figures are manipulable from a Word document in Windows with a right click (choose Object Cabri3DActiveDoc-Manipulate) if a plug-in is installed on the machine. A free plug-in for manipulating the figures is downloadable from www.cabri.com. It is also possible to open the corresponding Cabri 3D file by clicking on the link to the file (Ctrl+click) if Cabri 3D is installed on the machine. A free evaluation version is also available from the same site.

The student will count faces in front of him/her, try to memorize the already counted faces and rotate the polyhedron to count other faces but very soon will be lost as all faces look the same. The situation would be completely different if the student would have first been introduced to the genesis of this rhombic polyhedron from a cube: on each face of the cube is added a pyramid with vertex the reflected image of the center of the cube with respect to this face. Or in other words, each of the six inner pyramids with vertex the center of the cube which build the cube is turned inside out (Fig.2).

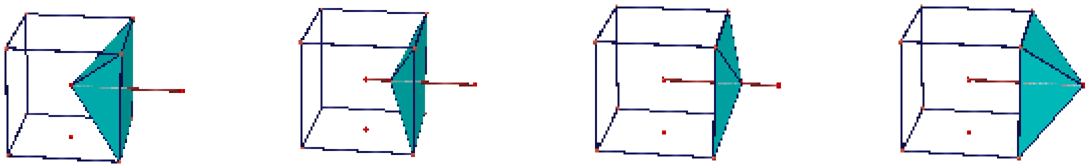


Figure 2 – Turning inside out a pyramid pyramidincube0.cg3

When looking at the obtained polyhedron, one can recognize that two triangular faces sharing an edge of the cube are in the same plane (Fig.3) and seem to build a rhombus with one diagonal being an edge of the cube.

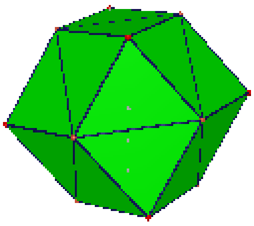


Figure 3 – Two faces sharing an edge of the cube rhombicdodecahedronwithcube.cg3

At this point, the counting of the number of the faces of the new polyhedron can be assisted by the structure coming from its genesis. Each of the six faces of the cube gives rise to four triangular faces and as each face of the polyhedron is made of two triangular faces, the number of its faces is 6 x 4 : 2 i.e. 12. Another counting process is based on the fact that to each edge of the cube corresponds one face of the polyhedron this edge is diagonal of. As the cube has 12 edges, the polyhedron has 12 faces.

This example intends to illustrate how counting the faces of a polyhedron requires going beyond the visual perception of the faces of the polyhedron and structuring this polyhedron. Two examples of possible structures were given here for the rhombic dodecahedron. This is why enumerating tasks are used in teaching as powerful tasks for fostering the construction of a mental structure for solid objects. Similar experiments with other solid objects (pyramids, prisms) carried out with students at the end of high school provided the same evidence (Mariotti 1992, 1996). Constructing a structure for solid objects requires identifying the parts of this object of same or lower dimension and their mutual relationships.

2.2. Breaking down a geometric figure into components

The complaint of many teachers about difficulties students encounter in “visualizing” 3D objects or figures, as well as the emphasis made by some curricula on the importance of learning to “visualize” in 3D geometry, reflect the same idea of the absence of interpretation in geometrical terms by students of what they see. As written by Gutiérrez (1996, p.9) visualization refers to “the kind of reasoning activity based on the use of visual and spatial elements, either mental or physical, performed to solve problems or prove properties". He considers mental images or cognitive representations of mathematical properties as a main element of visualization, actually shared by different theoretical approaches about visualization. Mental images differ from visual perception even if they may derive from it. For example, Kosslyn (1994, p.329) claims that images contain previously digested information; they are organized into units that have been previously interpreted whereas perception organizes the input from scratch and match it to stored representations. The importance of the role of mental images in mathematics is shared by constructivist approaches (as claimed by Presmeg in her review of research about visualization, 2006).

More recently, the same claim is expressed by Duval (2005) who distinguishes between two ways of "seeing" a figure in 2D geometry or 3D geometry:

- An iconic visualization bearing on the shape: a child recognizes a round shape in a disc or in a circle and is able to distinguish it from a squared shape; the shape of a ball is also easily distinguished from the shape of a cube. The criterion for recognizing the shape bears on the contour of the global object. Shapes must be stable for being recognized;
- a non iconic visualization in which the figure is broken up into components or is transformed into another figure.

In Duval terms, the iconic visualization of a cubic box does not consider the relationships between faces, edges and vertices of this box. A strong evidence of this is the well-known difficulty children encounter when counting the number of faces or of edges of a cube even when the cube is available for manipulation. As long as the children do not have structured the cube into for instance the top and bottom faces and lateral faces, they must have recourse to marks on the material cube in order to memorize the faces already counted.

In a non iconic visualization, a cube can be broken down

- into a set of polyhedrons, like three congruent pyramids with a common vertex and a squared base or two prisms on a triangular base (Fig.4);
- into a set of faces: it can be considered as a prism constructed on a squared face
- into a set of edges: four systems of three edges orthogonal to each other and sharing a common endpoint (points 1, 2, 3 and 4 on Fig.5);
- into a combination mixing edges and faces: it can be structured as made of two parallel squared faces connected by four edges perpendicular to the faces (Fig.6).

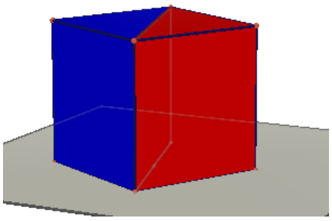


Figure 4 - Two prisms

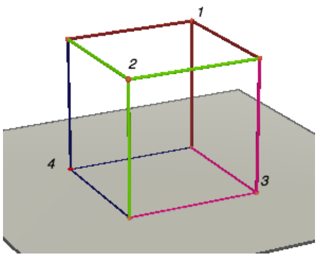


Figure 5 - Four systems of three edges

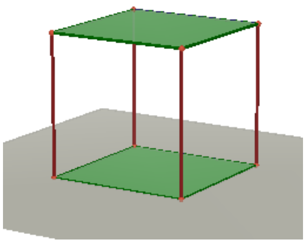


Figure 6 - Two faces connected by four edges

According to the problem to be solved, one way of breaking down the cube is more appropriate than the other ones. This cognitive process of splitting up an object into subparts of same or lower dimension is the core of the non-iconic visualization and is required in any problem solving in geometry (Duval 2005). This process can be supported by adding some elements on the diagram and/or hiding other elements. The visualization of a cube as made of two prisms is more apparent when the common rectangle of both prisms is drawn.

Although geometry requires both types of visualization, the non iconic visualization is essential for identifying and reasoning about geometrical properties. The non iconic visualization must be learned. This is not an easy task as the iconic visualization which is immediate may sometimes hinder the non iconic visualization. The recognition of a prism is much easier for students when the base is horizontal. The prisms of the deconstruction of the cube (mentioned above) usually are not seen by students on a diagram if their base is not in a horizontal or vertical plane (Fig.7). The iconic visualization when the base is not horizontal or not vertical is more focusing on the ‘corners’ of the prism than on its parallel edges.

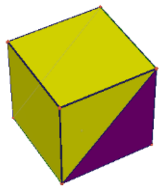


Figure 7 - Prisms in a non prototypical position

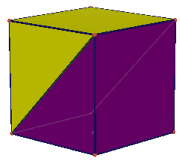


Figure 8 - Prisms in a prototypical position

The possibility of manipulating the cube to move the prisms in a prototypical position helps students see the half cube as a prism (Fig.8). This manipulation allows students to eliminate the conflict between the iconic and the non iconic visualization.

Too often the teaching of mathematics ignores that students have not yet constructed a non iconic visualization and does not help students be able to develop it. Fishbein (1993) developed the notion of figural concept to give account for this dual role of figural and conceptual in geometry.

3. Graphical and textual registers

Each representation of a mathematical object brings some aspects to the fore, whereas it hides other aspects of the same object and thus affects the way the object is conceived. The meaning constructed by the individual is not only affected by the features of the representations available but also by the possible ways to use them. Mathematical activity requires manipulations of and operations on these representations. Various systems of representation in mathematics have been built over time, and these systems affect how we do mathematics. Netz (1999) argues that Greek mathematics was both supported and limited by the available media. Kaput (2001) claims that fundamental representational infrastructures, such as writing systems and algebra, play a major role in determining what and how people think and what they are capable of doing.

Learning mathematics and learning to have a mathematical activity require being able to choose the adequate register for the problem to be solved and possibly to move to another register. What we meant is that the flexibility of moving between registers is not only supporting the construction of the meaning of a mathematical concept but is essential when «doing» mathematics. Duval (2000, pp.1.63-1.65) claims that understanding a concept requires coordinating at least two registers and being able to move spontaneously and rapidly from one register to another one. In geometry, two registers are indispensable: the graphical register of diagrams and the textual register. As a geometrical figure cannot be entirely determined only from its diagram, a textual description specifying the objects and relationships determining the figure is needed (Parzysz 1988).

It has often been stated that the difficulty of proving in geometry for students lies in the subtle role of diagrams in the elaboration of the proof. On the one hand, diagrams provide ideas about how to justify a statement and it would be impossible to write down the proof of a complex problem without a diagram (Laborde 2005). On the other hand, a proof cannot include elements coming from visual evidence. This subtle role of diagram is far from being used spontaneously by students. The role of the teacher is essential in making students more familiar with this game.

We believe that new 3D geometry environments offer useful tools that can be used by the teacher for the development of both a non-iconic visualization and flexibility between diagrams and texts. This claim will be illustrated in what follows by means of Cabri 3D.

4. 3D dynamic geometry environments with direct manipulation

4.1. Amplifying the reliability of iconic visualization

One of the problems of 3D geometry is that 3D objects can be represented only in 2D even on computer screen unless these objects are represented by material solid objects (such as mock-ups). In 2D geometry, the iconic visualization could hinder the recourse to non iconic visualization but the evidence given by iconic visualization is generally reliable. It is no longer the case in 3D: it is not possible to be sure that two lines intersect from a diagram, or that four points or more are coplanar. It was often observed by teachers that middle school or high school students believe that two lines intersect in 3D because they intersect on the diagram.

The possibility of changing the point of view in 3D dynamic environments with direct manipulation allows the user to obtain immediate visual evidence of such phenomena, as in the following Cabri 3D figures of two non intersecting lines (Fig.9) or of four non planar points (Fig.10). In 3D dynamic geometry environments, changing the point of view so that three points seem to be on a line provides iconic evidence whether four points are coplanar or not: if the fourth point seems to be outside of the line, the four points are not coplanar.

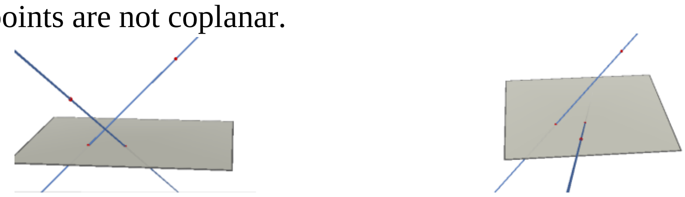


Figure 9 - Two apparently intersecting lines seem no longer intersect twoapparentlycrossinglines.cg3

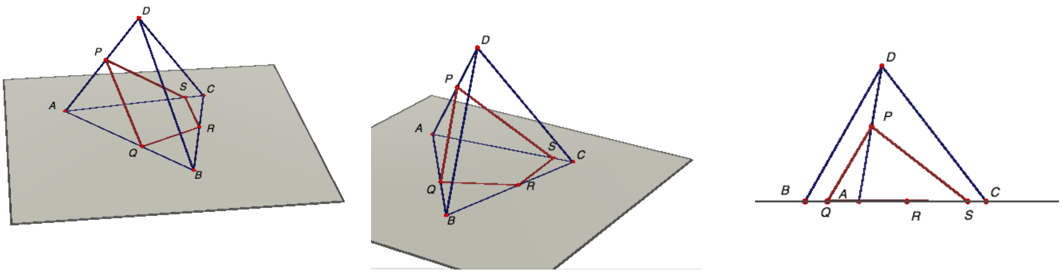


Figure 10 - Four points from different points of view fourpointscoplanarornot.cg3

In the same manner, the possibility of unfolding and folding of any variable polyhedron amplifies the possibilities of visual evidence that in absence of technology are obtained only at the cost of making a material net for each instance of solid objects.

The continuity of the move in all the above examples plays a critical role on the fact that the individual considers that it is always the same object (s)he is viewing.

4.2. Assisting the development of non iconic visualization: the role of construction tasks

Facing students with appropriate tasks to be solved in a 3D dynamic geometry environment may be the source of the development of non iconic visualization. Construction tasks seem to be well suited for such development as in most cases the construction of a complex object can only be done in breaking down the object into parts and in constructing these parts by taking into account their mutual relationships. Construction tasks require a cognitive process of "deconstruction" of the complex object to construct and coordination of the units obtained in the deconstruction process. More precisely, a construction task requires the following processes:

- breaking down the object into units;
- identifying the geometrical nature of the units: these parts may be points (dimension 0), segments or lines (dimension1), planes or polygons (dimension 2) or solid objects (dimension 3, prisms, pyramids, cubes);
- for each unit to reconstruct, identifying its relationships with the already reconstructed units so that it can be determined.

The novelty of 3D geometry environments is that this deconstruction is not only possible with 0 or 1 dimensional parts like on paper and pencil, but also with 2 or 3 dimensional parts as far as the software environment offers tools for constructing 2D and 3D objects. Construction tasks of 3D objects in those environments may call thus for an analysis of 3D objects focusing on components of dimension 2 or 3 and may contribute to a better knowledge of space. Before the availability of 3D geometry computer environments, construction by means of 2 or 3 dimensional parts were only possible with mock ups or games.

The very simple example of the cube will be used to illustrate this claim. The teacher asks the students to get rid of the tool « Cube[2]» and to construct a cube from a given square in the base plane. The most spontaneous strategy from students is to construct the edges of the cube and not the faces. They do it by working in the planes of the lateral faces obtained as perpendicular planes containing an edge of the starting square. They construct squares in each plane by using circle. This dimensional deconstruction structures the cube as a net of edges and consists in coming back to construction of squares in planes. This is the most usual strategy for students as they are mainly familiar with 2D geometry.

We also had experimented this task in in-service teacher education and could observe the same strategies based on reconstructing the cube from edges. When prompted to use faces (2 dimensional objects) most of teachers used planes: the cube was obtained as intersection of a set of parallel and perpendicular planes (Fig.11).

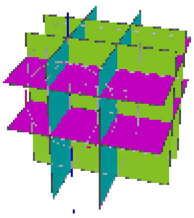


Figure 11 – A cube as intersection of planes cubeasintersectionofplanes.cg3

[2] This possibility of customizing the toolbar is available in Cabri 3D v.2.1.1

They did not resort to transformations which are efficient construction tools available in Cabri 3D. For example, a lateral square can be considered as the image of the base in a rotation with axis a side of the square in the base plane (Fig.12). The other lateral faces can be obtained as images of the previous one in rotations around the vertical axis of the cube (Fig13).

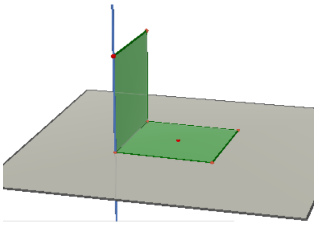


Figure 12 – A face rotated from the initial square

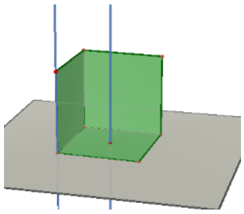


Figure 13 - Another lateral face rotated from the previous one

In experimenting reconstruction tasks of a prism or of a complex object made of several prisms with three pairs of 10th French graders (15 to 16 year-old students) who have been just introduced to Cabri 3D before the experiment, Mithalal (2007) also observed that:

- students preferred to reconstruct units by means of points instead of segments or lines
- even if they were able to use transformations in a plane, they did not use them in space even if they identified them.

For example, a prism (Fig.14) was given on the screen of Cabri 3D and students had to reconstruct an identical object with same behavior on another computer without having available in Cabri the prism and polyhedron tools. This prism was constructed from three directly movable points, points *a* and *b* in the base plane and point *c* on the line perpendicular at the base plane in point *O* (center of the bottom face of the prism).The three pairs did not recognize a prism or at least did not pronounce this word. All the pairs considered the prism as made of one bottom face, one top face and vertical edges in order to reconstruct it. The bottom face in the base plane was identified as a rhombus by all students. Two pairs of students used the symmetry with respect to point *O* to reconstruct the bottom face and used the tool central symmetry on the vertices of the rhombus and not on its sides. Pair 1 identified that the top face was symmetrical with respect to point *c* and reconstructed the vertices of the top face as images of the vertices of the bottom rhombus in this central symmetry.

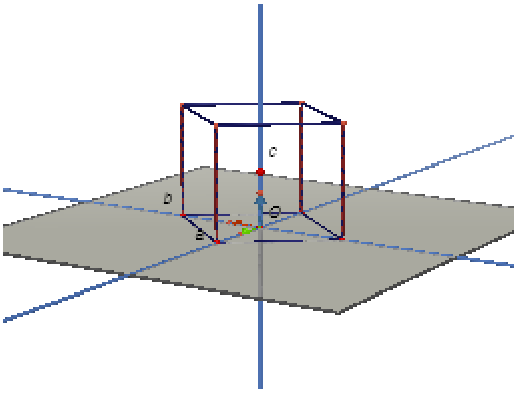


Figure 14 – The prism to reconstruct prism1.cg3

Pair 2 was very puzzled when having to construct the top face. S1 and S2 denote here the students of this pair.

S1: “You must have a second face, indeed at the top! But I don’t know what to tell you to do. Indeed I don’t manage to understand how to do.” (Il faut que t’aies une deuxième face, en fait, en haut! Mais je sais pas comment te faire faire. . . En fait j’arrive pas à comprendre comment faut faire.).
In describing to his mate S2 the top face, student S1 explained with gestures that there were two identical rhombuses, the one at the bottom, the other one at the top. S2 thought that the plane of the top rhombus was passing through *c*, (probably because moving *c* provoked the move of the top face of the prism). S1 rejected this proposal saying that the top face was not at the level of point *c* since point *c* was in the middle of the figure (she meant in the middle of the prism). This intervention led S2 to identify symmetry.

S2: Point *c*, what is it? (Le point *c*, c’est quoi ?)
S1: Point *c*, it is the point… (Le point *c* c’est le point. . .)
S2: It is the central point of the other face, of the top face! (C’est le point central de l’autre face, de la face du dessus !)
S1: No (non)
S2: Well, what is it? (ben c’est quoi ?)
S1: It is in the middle of the figure (il est au milieu de la figure)
S2: So well it is the central point of the top face (ben oui, donc c’est le point central de la face du dessu)s
S1: No, it is in the middle of the figure (non, il est au milieu de la figure)
S2: I got it, I think that I found, you said that point *c* is in the middle of the figure, it means that I do the symmetry of that point (Ah, mais alors j’ai compris ! je crois que j’ai trouvé! T’as dit que le point *c* il est au milieu de la figure, ça veut dire que je fais la symétrie de ce point là!).

They constructed the reflected image *O’* of point *O* with respect to point *c* but did not resort to symmetry to construct the images of the vertices of the bottom rhombus. They constructed the vertices of the top rhombus as intersection of parallel lines to lines *Oa* and *Ob* passing through *O’* and perpendicular lines to the base plane passing through the vertices of the bottom rhombus. It seems that this pair first had difficulties to extend the notion of symmetry from plane to space. Firstly they identified with difficulty a symmetry in space although they used it without any problem in the base plane but the fact that this latter is horizontal may have facilitated the use of a symmetry. Secondly once they have identified a symmetry, they reduced its use to one point and did not use it for other points of the top face of the prism. They could have used symmetry in two different ways: either in space with respect to point *c*, or in the plane of the top face with respect to *O’* but they did not. The properties prevailing over the central symmetry were parallelism and perpendicularity between lines. The vertical direction of lines may have reinforced the use of the perpendicularity.

Pair 1 used other transformations in the following reconstruction tasks (Fig. 15, 16 and 17), a central symmetry only for points in the second task and then axial and plane symmetries for segments of the two last tasks as they became aware that using transformations directly on segments was faster than using it for points and then constructing the segments joining points. The repetition of the construction tasks led the students of pair 1 to be aware that it is more efficient to use directly a transformation on edges than on points. It confirms our assumption that tasks may favor the evolution of the strategies of students, and in this case the evolution of the instrumental genesis (see section 4.4).

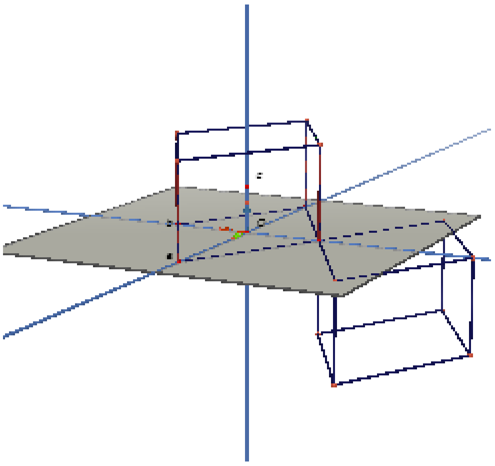


Figure 15 – Reconstruction task 2 prisms2.cg3

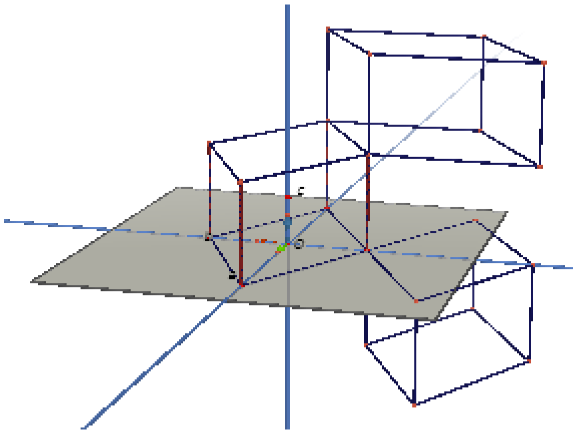


Figure 16 - Reconstruction task 3 prisms3.cg3

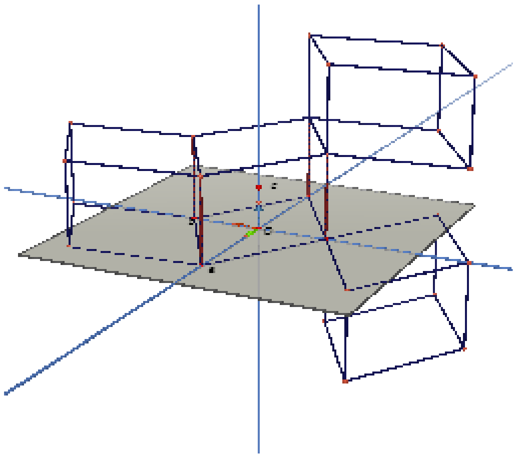


Figure 17 - Reconstruction task 4 prisms4.cg3

The observation of pair 2 on the other tasks confirms their difficulty of using a symmetry in space. They constructed the top face of the second prism (Fig.15) by using a central symmetry as this face was in the horizontal base plane but then resorted to the use of perpendicularity and parallelism for completing the construction. They constructed the vertices of the bottom face of the third prism as images of the vertices of the top face of the first prism through an axial symmetry, then constructed the reflected image of point *O* with respect to the center of the top face of the first prism to determine the height of the center of the top face of the third prism and then completed the construction by using parallel lines and lines perpendicular to the base plane (Fig.16).

Pair 3 also used a central symmetry only in horizontal planes and on points for the construction of the rhombi. At the beginning pair 3 produced mainly figures by means of measures adjusted through dragging. As such dragging did not preserve the apparent shape of the figures, the students moved then to the use of central symmetry on points and the construction of parallel lines.

From all these observations of students and teachers faced with construction tasks with Cabri 3D, some conclusions can be drawn:

- in the analysis phase of the figure to reconstruct
    - plane figures, in particular polygons are easily identified

  - basic properties such as parallel or orthogonal directions are identified, especially when directions are vertical or in an horizontal plane
- in the reconstruction phase
  - the use of properties is preferred to the use of transformations in space which is not spontaneous
  - if transformations are used, they are mostly used on points and not on objects of higher dimension; however this can change either under the suggestion by the teacher or from the awareness of the tedious character of a construction made only on points.

The analysis phase seems to show that the structure of a complex object is mainly done according to horizontal planes and vertical lines. The reconstruction phase seems to show that plane geometry interferes strongly in the reconstruction process. Transformations are used within planes and particularly horizontal planes and not extended to space. We hypothesize that plane geometry affects the ways of structuring 3D space. I addition to students' difficulty of anticipating the effect of a transformation in space, their scarce use of transformations and in most cases dealing only with points may also be explained by the influence of paper and pencil geometry in which constructions are done by using geometric properties and in which it is not possible to obtain directly the images of a figure through a transformation. When faced with a new environment, students must learn how to use it and, beyond knowing how to use technically the tools offered by environment, they also must be able to develop new construction strategies based on these tools. This point will be developed in section 4.4.

To conclude this section, we stress the potentialities of construction tasks in Cabri 3D like computer environments for non iconic visualization, as new ways of breaking down complex figures are made possible by these environments and new construction tools based on transformations are made possible. Feedback offered by dragging may allow students to invalidate some of his/her incorrect construction strategies and lead them to seek another solution. Of course, facing students with only one such task is not enough to foster such changes in their strategies and only the combination of a sequence of tasks and of teacher interventions can be the source of changes.

4.3. The importance of available tools

The use of a tool affects the way a subject solves problems depending on the actions made possible by the tool. However in the first uses of a tool, very often the schemes of utilization developed by the subject are not the most efficient, as seen above in the preceding section. The tool becomes an efficient instrument for the subject only after a process of 'instrumental genesis' (Rabardel 1995), in which the teacher can play a critical role. The teacher gives tasks for which some specific software tools offer efficient solving means. Even if the students solve the tasks not by making use of these efficient tools and by resorting to more tedious strategies, they can appreciate the power of the tools presented by the teacher after they attempt to solve the task. As tools in 3D geometry software are strongly related to mathematical properties and objects, students construct knowledge not only about tools but also about mathematics.

Very relevant and beautiful examples of original and efficient instrumentation with Cabri 3D are provided by Chuan (2006)[3]. They come from the lecture given by Chuan at ATCM 2006 entitled

[3] at the address sylvester.math.nthu.edu.tw/ d2/talk-atcm2006-unmotivated/

“Some unmotivated Cabri 3D constructions”. “Unmotivated” was explained by Chuan as “non algebra, non routine, not found in Euclid, discovered accidentally, tailor made, so short, so beautiful, so fun”. For most all these reasons, we consider that facing students with these construction tasks is supporting students learning of a deeper non iconic visualization and thus a better knowledge of space geometry. These constructions are non routine and not found in Euclid because the tools they required were not available. Chuan insists on the efficiency of the constructions (short). This is a critical feature of problems that are able to promote learning of new knowledge according to the theory of didactic situations (Brousseau 1998). A new solving strategy is likely to be constructed by an individual when his/her routine or available strategies are tedious or inoperative for the problem. The beauty of the solution emerges from the conjunction of its efficiency and its unusual character. Let us comment two examples given by Chuan: the construction of the triangular cupola starting from a hexagon and the construction of the square orthobicupola starting from a cube.

A triangular cupola is obtained from a cuboctahedron (Fig 18) cut by a plane containing six edges forming a hexagon (plane *ABC* on Fig. 18) whose center is the center of the cuboctahedron.

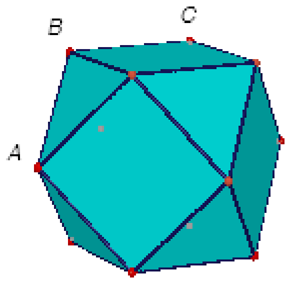


Figure 18 – A cuboctahedron [cuboctahedroncut.cg3]

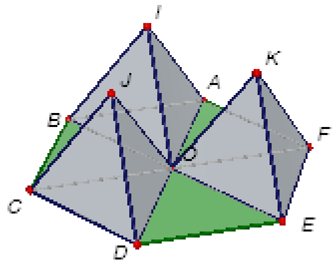


Figure 19 – Three tetrahedra [threetetrahedracupola.cg3]

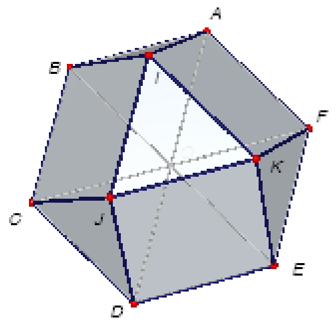


Figure 20 – Construction of the cupola as convex hull triangularcupola.cg3

All the vertices of the cupola are equidistant from the center of the hexagon and hence tetrahedron *OABI* is regular. The cupola can be obtained as the convex hull of three tetrahedra constructed on equilateral triangles *OAB*, *OCD* and *OEF* (Fig.19, Fig.20).

This solution is based on a deconstruction of the cupola into 3D and 2D components of the figure. Of course a more traditional deconstruction into 2D and 1D components could be carried out. Such a deconstruction could be used in a strategy based on the determination of vertex *I* as a vertex of equilateral triangle *OAI*. The two other vertices *J* and *K* of the triangular top of the cupola can be obtained as images in a rotation around the axis of the cupola. This would lead to a longer construction.

The square orthobicupola (Fig. 21) is a square cupola reflected with respect to its octagonal base. It can be obtained as the convex hull from five cubes. Start from a cube, rotate it around each edge of the top square with an angle of 45° (Fig. 22, Fig. 23). Then create the convex hull of the five cubes with the tool “Convex Polyhedron”.

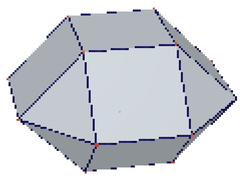
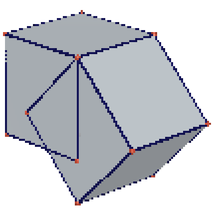


Figure 21 – A square orthobicupola squarebicupola.cg3



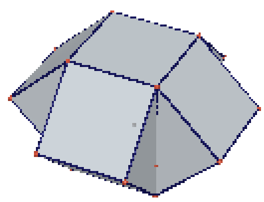


Figure 22 – Bicupola step2 cuberotated.cg3 **Figure 23** – Bicupola step 5 4cubesrotated.cg3

What is new, is the possibility of constructing a 3D object from other 3D objects. Earlier, this possibility dealt only with material objects and was often reserved to jigsaw recreational activities. From a learning point of view, this construction possibility by means of 3D objects is interesting because it relates 2D and 3D objects: in the triangular cupola, for example, it requires considering the lateral equilateral face of the cupola as a face of a regular tetrahedron, in the orthobicupola it requires considering a lateral square as the face of a cube rotated from the central cube. The example of the square orthobicupola differs from the example of the triangular cupola in that the 3D parts from which the convex hull is built are non disjoint and as such differs from a construction with material objects.

The available tools of the environment affect very much the possible construction strategies. We have just seen how the availability of the tool providing the convex polyhedron hull of objects may be an efficient construction tool. Constructing a convex polyhedron requires thus identifying the minimal number of elements of lower dimension determining the polyhedron.

There are other geometric tools which can be used for a construction. A sphere is also a tool offering a transfer of measure from a point in any direction. The transfer of a measure in a plane from a point is also possible by using a circle around an axis (Fig. 24). Transferring the length of a segment on a perpendicular line from one of its endpoint can be done in the same way by using a square (Fig.25). The transfer of angles of some regular polygons is also possible by using the tools regular polygons around a line or a segment (Fig.26). All these tools are based on geometric properties of regular polygons and of circles.

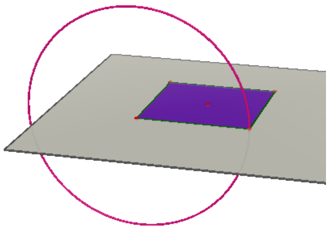
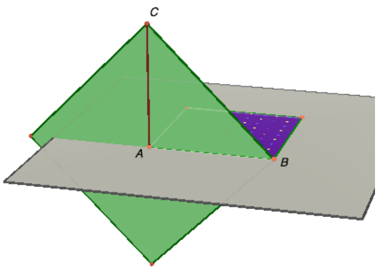


Figure 24 – …a circle



Length transfer by means of
Figure 25 – …a square

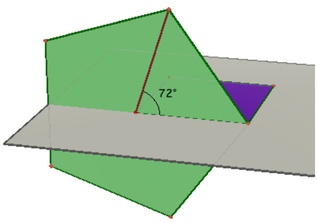


Figure 26- Transfer of an angle

The non iconic visualization in a length transfer is called twice

- in identifying that on the 3D figure to be constructed, two segments (elements of dimension 1) are congruent
- in adding another object of dimension 2 or 3 in which two segments with the same relative position are also congruent.

A process of going down and up in the dimensions of the objects is very much involved in the use of these tools. Therefore we consider that construction tasks in 3D geometry environments offering this kind of tools is very demanding in terms of geometric knowledge and conversely can be used by teachers to promote the development of non iconic visualization in 3D.

Transformations offered by Cabri 3D are also tools that can be used to construct 3D complex objects. Identifying that one part of the object is the image of another part in a transformation is also a matter of non iconic visualization.

The interface representing continuously the image in the construction process of a rotation around an axis when the angle is increased continuously from 0 until its target value provides iconic visualization simulating a real motion in space and hence supports the non iconic visualization. We assume that this possibility of seeing a continuous movement between an object and its image can be used by the teacher to provide imagery of movements in space that very often students do not have and to relate these movements to geometrical transformations. Students can be asked to simulate a triangle rotating around one of its sides, or a face of a polyhedron to rotate around one of its edges, or a polyhedron around an axis (Fig.27).

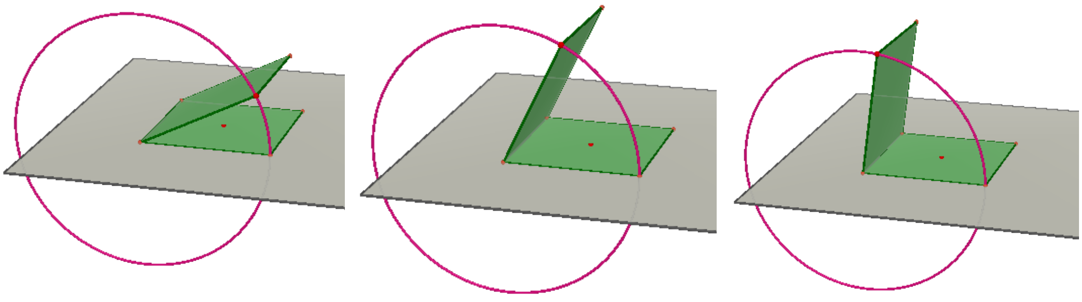


Figure 27 – A rotating square around an edge of another square rotating%20square.cg3

4.4. Complementary roles of graphical and textual registers

Let us come back to the example of four points apparently coplanar. As said above, changing the point of view allows the students to augment their iconic visualization and to invalidate the fact that the four points are coplanar. However the reason why four points are coplanar or non coplanar can only be found by using theoretical knowledge. The textual description of a figure offered by Cabri 3D describes the objects and their relationships (geometrical properties) that define this figure (Fig.28).

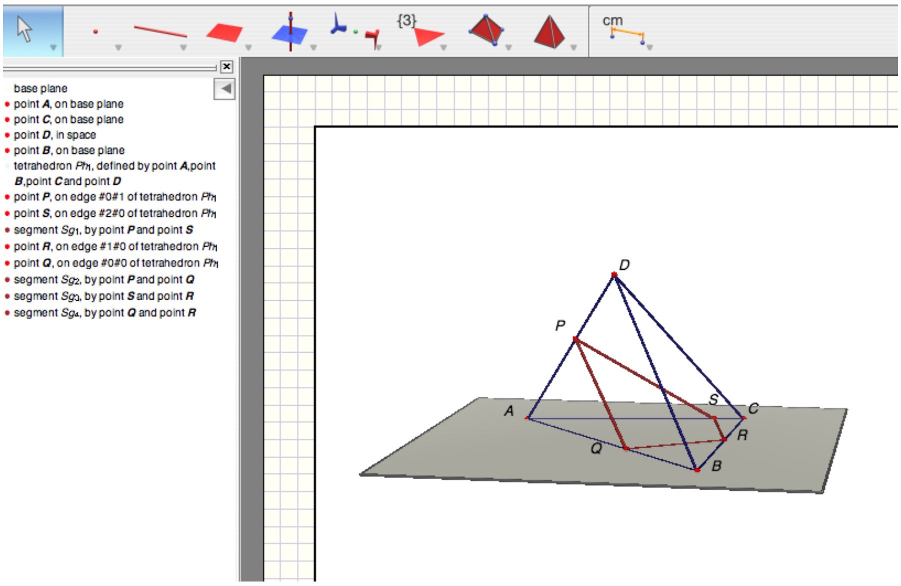


Figure 28 – A figure and its textual description fourpointscoplanarornotwithdesc.cg3

The surprise of the students discovering that four points they expected to be coplanar are not coplanar, can be used by the teacher to motivate students to prove why. Students often have

difficulties in not using evidence given by the figure in their proof: it is clear that three points are in the same plane and the fourth one is not. The existence of the textual description can be utilized by the teacher: the proof of the fact that these four points are not coplanar must be based only on information items given by the textual description.
The teacher can ask: Why do we know from the description that R, Q and S are in the same plane? Why do we know from the description that P is not in this plane? The link between the textual description and the figure can help students find in the description all information items about an object. When clicking on the object in the diagram, all the occurrences of the object in the textual description are highlighted and conversely when clicking on an object of the description, all the other occurrences of the same object are highlighted and its representation in the diagram is flashing. While the description provides an objective criterion for insuring that the proof is only based on properties used to build the figure, the interactive link between text and diagram allows the student to reason by using non iconic visualization coming from the diagram. It offers a way of overcoming the paradoxical situation which students face: they must elaborate a proof with the help of the diagram but are not allowed to refer to the diagram in the text of the proof.

Such a proof requires to make explicit some theorems and axioms of 3D geometry. This can be used to make university students, in particular pre-service teachers, aware of the axiomatic system of geometry. Cabri II on the TI 92 was already used in this way to introduce university students to the axiomatic system of geometry and to do formal proofs (Perry Carrasco et al. 2006) or Italian high school students to construct a system of axioms (Mariotti 2000).

Teachers often consider that 3D geometry is hard matter to learn. In this paper, we pointed out two cognitive processes contributing to the difficulty of 3D geometry: iconic visualization and non iconic visualization. These processes are an essential part of any geometrical activity. We attempted to show that tools available in new 3D dynamic geometry environments may not only assist these cognitive processes but enlarge their range. Since it makes accessible operations on 2D and 3D objects, it may extend non iconic visualization to those objects. Of course the teacher is still needed. One of his/her roles is to design challenging tasks requiring an extended non iconic visualization. Such tasks can even be fun when, for example, they consist in reproducing dynamic 3D objects given on the screen of the computer as in Chuan's examples.

References

[1] Brousseau, G. (1998).*Theory of didactical situations*, Dordrecht: Kluwer Academic Publishers

[2] Chuan, J. (2006). Some unmotivated Cabri 3D constructions, Lecture at ATCM 2006, Available: http://sylvester.math.nthu.edu.tw/d2/talk-atcm2006-unmotivated/ (Accessible December 27, 2007).

[3] Duval, R. (2000). Basic issues for research in mathematics education, In T. Nakahara, M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 55-69) Hiroshima: Hiroshima University.

[4] Duval, R. (2005). Les conditions cognitives de l'apprentissage de la géométrie : développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements, In: *Annales de didactique et sciences cognitives*, 5-53, Strasbourg, France: IREM, Université Louis Pasteur.

[5] Fishbein, E. (1993). The theory of figural concepts, *Educational Studies in Mathematics*, *24*(2), 139-162.

[6] Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: In search of a framework. In L. Puig & A. Gutérriez (Eds.), *Proceedings of the 20th International Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 1, pp. 3-19, Valencia Spain: Universidad de Valencia

[7] Kaput, J. (2001, January). *Changing representational infrastructures changes most everything: The case of SimCalc algebra, and calculus*. Paper presented at the NAS symposium on improving learning with informational technology. Washington, DC. Available: http://www.simcalc.umassd.edu/library/library_2001.php. (Accessible September 20, 2007).

[8] Kosslyn, S.M. (1994). *Image and Brain. The Resolution of the Imagery Debate.* Cambridge, Mass.: The MIT Press.

[9] Laborde C. (2005). The hidden role of diagrams in students' construction of meaning in geometry In J. Kilpatrick, C. Hoyles and O. Skovsmose (Eds.), *Meaning in mathematics education* (pp.1–21). (pp.159-180) Dordrecht : Kluwer Academic Publishers

[10] Mariotti A. (1992). Le raisonnement géométrique : une dialectique entre les aspects figural et conceptuel, *Topologie structurale*, 18, 9-17.

[11] Mariotti, A. (1996). *The interaction between images and concepts in geometrical reasoning* PhD dissertation of the Tel Aviv University.

[12] Mariotti, A. (2000). Introduction to proof : the mediation of a dynamic software environment. *Educational Studies in Mathematics, 44 (1-3)*, 25–53.

[13] Mithalal, J. (2007). Changement du statut des objets en géométrie par le travail de la géométrie de l'espace, Mémoire de master 2 « Environnements Informatiques d'Apprentissage Humain et Didactique », Université Joseph Fourier, Grenoble

[14] Netz, R. (1999). *The shaping of deduction in Greek mathematics*. Cambridge, UK: Cambridge University Press.

[15] Parzysz, B. (1988). Knowing vs seeing: Problems of the plane representation of space geometry figures. Educational Studies in Mathematics 19(1), 79-92.

[16] Perry Carrasco, P., Camargo Uribe, L., Samper, C. Rojas Morales, C. (2006). *Actividad demostrativa en la formacion inicial de profesores de matematicas*. Bogotá: Universidad Pedagógica Nacional, 2006. 363 p.; incl. ref. 2006 Impreso Es.

[17] Presmeg, N. (2006). Research on visualization in the learning and teaching of mathematics, Emergence from Psychology, In A. Gutiérrez and P. Boero (Eds.) *Handbook of Research on the Psychology of Mathematics Education* (pp.205-235), Rotterdam: Sense Publishers

[18] Rabardel, P. (1995). Les hommes et les technologies. Paris: Armand ColinG